Scaling with respect to disorder in time-to-failure

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We revisit a simple dynamical model of rupture in random media with long-range elasticity to test whether rupture can be seen as a first-order or a critical transition. We find a clear scaling of the macroscopic modulus as a function of time-to-rupture and of the amplitude of the disorder, which allows us to collapse neatly the numerical simulations over more than five decades in time and more than one decade in disorder amplitude onto a single master curve. We thus conclude that, at least in this model, dynamical rupture in systems with long-range elasticity is a genuine critical phenomenon occurring as soon as the disorder is non-vanishing.

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I. SMALL VERSUS LARGE HETEROGENEITY IN SYSTEMS WITH LIMITED LOAD TRANSFER

Materials break down according to two broadly defined scenarios. In the first one, examplified by a pure crystal, there is no or little damage up to the rupture which occurs suddenly with no appreciable precursors. In the second scenario that applies ideally in the limit of very heterogeneous media, the system is progressively damaged, first in an uncorrelated way reflecting the pre-existing heterogeneity. As stress or strain increases, the damage becomes more and more correlated with crack growth and fusion, announcing the incipient rupture. This second regime is like percolation at the beginning and correlated percolation later and at the end of the process. In the limit of infinite disorder, the rupture (in the quasi-static limit) can actually be mapped exactly onto a percolation problem [1]. This second scenario is characterized by a growing susceptibility and well-defined precursors. This classification has been emphasized by Mogi [2] in his search for earthquake precursory phenomena. Basing his reasoning on an analogy between elastic shocks (acoustic emissions) caused by fracture in heterogeneous materials [3] and earthquakes, he noticed that the fracture process strongly depends on the degree of heterogeneity of materials: the more heterogeneous, the more warnings one gets; the more perfect, the more treacherous is the rupture. The failure of perfect crystals is thus unpredictable while the fracture of dirty and deteriorated materials could be forecasted. For once, complex systems could be simpler to apprehend! However, since its inception, this idea has not been much developed because it is hard to quantify the degrees of "useful" heterogeneity, which probably depend on other factors such as the nature of the stress field and boundary conditions, the presence of chemical contaminants, etc. This analogy nevertheless led Mogi to hope that earthquake precursors could be identified for forecasting. Nowadays, finding clear earthquake precursors is an active and controversial research domain whose ultimate objective is still eluding the scientific community [4].

In contrast, the situation is more favorable in the laboratory and in models. For systems where the load transfer has limited stress amplification, the situation has been clarified. By limited stress amplification, we refer to the cases where the stress transfer due to a crack or more generally a damage zone does not exhibit spatial concentration. This can occur in a variety of ways, for instance, in the democratic fiber bundle model [5] with the democratic rule of stress transfer, or in models of block-springs with stick-slip behavior induced by solid friction in which the stress transfer is screened beyond the sticking blocks [6]. In real systems, fiber-matrix composites exhibit this property as a local fiber rupture is locally accomodated by a distorsion of the matrix which screens any local concentration and help uniformize the stress [7]. For such systems, it has been shown recently [8] that disorder is a relevant field leading to tri–criticality, separating a first-order (or abrupt) regime where rupture occurs without significant precursors from a second-order (or continuous critical) regime where the macroscopic elastic coefficient exhibit power law behavior. These results have been obtained using analytical solutions of fiber bundle models and numerical simulations of a two-dimensional tensorial spring-block system in which stick-slip motion and fracture compete [8]. The idea is that, upon loading a heterogeneous material, single isolated microcracks appear and then, with the increase of load or time of loading, they

both grow and multiply leading to an increase of the number of cracks. As a consequence, microcracks begin to merge until a "critical density" of cracks is reached at which the main fracture is formed. It is then expected that various physical quantities (acoustic emission, elastic, transport, electric properties, etc.) will vary. However, the nature of this variation depends on the heterogeneity. The new classification uses the fact that there is a threshold that can be calculated : if disorder is too small, then the precursory signals are essentially absent and prediction is impossible. In the language of phase transitions, the heterogeneity is a control parameter (like the chemical potential in the Blume-Emery-Griffith Ising model) that controls the distance to a so-called tri-critical transition as the disorder increases, from a Griffith-type [1] abrupt rupture ("first-order") regime to a progressive damage ending at rupture, corresponding to a critical or "second-order" transition. The other control parameter controls the distance to global rupture and can be time, strain, stress or combinations of these. The transition between the two regimes, which are two modes of brittleness culminating in a sudden failure, is different from the brittle-ductile or brittle-plastic transitions. The value of the disorder threshold separating these two regimes depends on the system strength and other properties. This has in fact been tested carefully in several laboratory rupture experiments, for instance using acoustic emissions as precursors of rupture of fiber composites [10]. The idea that rupture can be "critical" is not new [11] but the classification of why, when and how much so is useful.

II. THE CASE OF LONG-RANGE ELASTICITY

The purpose of this note is to complement these results by studying a case where the stress load transfer has *no* limiting amplification in amplitude and range, as for instance in genuine elasticity. In this case, a fracture of length a produces a stress amplification factor proportional to $\sqrt{a}$ extending to large distances, typically with a r^{-d} decay in d dimensions for $r \gg a$, as a function of the distance r to the crack tip. In a large system, an arbitrary large crack can thus in principle produce an arbitrary large stress increment, which can thus overcome any rupture barrier created by the heterogeneities. We thus expect the rupture to belong to a different class than for bounded stress load transfer.

A rupture of size a generates a stress intensity factor equal to $\sigma_{drop}\sqrt{a}$, where σ_{drop} is the stress drop. Let us call $\Delta\sigma$ the typical amplitude of the stress barriers preventing the progression of cracks. Two cases appear.

- $\sigma_{drop}\sqrt{a} < \Delta\sigma$: the amplitude of stress enhancement is smaller than the quenched heterogeneity. The latter thus dominates and we expect an organization similar to that observed in the previous case of limited load transfer.
- $\sigma_{drop}\sqrt{a} > \Delta\sigma$: sufficiently large cracks will always create stress transfer larger than the pre-existing barriers. Beyond a characteristic nucleation size a^* given by $\sigma_{drop}\sqrt{a^*} \simeq \Delta\sigma$, cracks will not be stopped and will always break through the system.

This one-body argument seems to favor the idea that rupture in systems with long-range elasticity should be "first-order", i.e. of the Griffith type. As the deformation is building up, damage first grows progressively and one should witness an increasing cooperativity of crack growth and coalescence up to the critical size a^*, at which stage a different regime is switch on and the macroscopic rupture is triggered abruptly. This is similar to the situation occurring during the nucleation of a new phase in first-order phase transitions in which droplet fluctuations below the critical radius suddenly leave place to the unstable phase growth when a droplet reaches the critical size.

This scenario turns out to be wrong and the reason for this is that the "mean-field" argument leading to the characteristic nucleation size a^* neglects long-range interactions between many such characteristic cracks in a large system. Solvable models and numerical evidence indicates that the rupture is in fact critical, i.e. exhibits an increasing cooperativity of crack growth and coalescence up to the macroscopic rupture itself [12–15]. Ref. [15] sees breakdown in disordered media as a first-order transition, while they present ample evidence of scaling of the fraction of broken bonds and of the divergence of the characteristic size up to the macroscopic rupture. Their claim is based on the fact that the fraction of broken bonds has a discontinuity with a jump just at rupture. We think that this is confusing as it does not distinguish these results from the genuine abrupt behavior of Griffith rupture, with no significant precursors and not divergence of a characteristic length. A way to reconcile these points of view is to notice that critical phenomena in random systems have in general more than a single correlation length. In rupture, one correlation length is the

[1] The Griffith criterion for rupture takes exactly the form of a condition for a critical "droplet" to nucleate and trigger the growth of the new phase [9].

characteristic size of the damage, which diverges upon appraoching the global rupture. A second charateristic length is the size of the system that dictates the size of the macroscopic fracture. We also stress that the power law scaling found in rupture in random media does not correspond to some sort of "spinodal" analog of a first-order transition. Indeed, "spinodal" decomposition corresponds to the long-time coarsening of the phase while rupture deals with the time behavior close to a critical control parameter value. We also note that, in elastic-plastic transitions in heterogeneous systems, there is not jump and the transition qualifies as critical [16]. Since a model of rupture in random media can be mapped onto this elastic-plastic transition [17], this further substantiates the idea that rupture in sufficiently heterogeneous media has the properties of a critical phenomenon.

These results have been obtained for sufficiently large disorder. The question arises whether the critical nature of rupture in the presence of long-range elasticity survives for all non-vanishing disorder or if rupture becomes truly abrupt for sufficiently weak disorder. Indeed, we know that, for exactly zero disorder, rupture is abrupt with no precursors or any diverging characteristic length : in fiber models of instance, identical fibers all break simultaneously when the stress reaches their common threshold. Our results below show that the subtlety, richness and complexity of rupture processes in heterogeneous media may be seen [8] to arise from the non-commutation of the two limits $(q \to \infty, \Delta \to 0)$, where q is the order of the moment $\langle \sigma^q \rangle$ of the stress distribution and Δ is the amount of disorder, in the same way that the non-commutativity of limits is at the crux of some of the major outstanding problems in physics [18] such as turbulence (viscosity $\to 0$; time $\to \infty$) and quantum chaos ($h \to 0$; time $\to \infty$). More practically, this corresponds to the non-commutation of the limits $(p \to p_r, \Delta \to 0)$, where $p_r - p$ is the parameter measuring the distance from rupture. p can be the time, stress, or strain depending on the system and boundary conditions. We now analyze carefully how the response of a system going to rupture depends on the amplitude of disorder.

III. MODEL AND RESULTS

We study the thermal fuse model [14] which has recently received experimental scrutiny [19]. Let us recall briefly its definition that will be useful for the discussion below. Fuses are put on the bonds of a square lattice of size $L \times L$. A fixed current I is imposed at time $t = 0$ accross the two-dimensional lattice. The fuses are heated by a generalized Joule effect (electric power $\sim$ (current)^b)

$$C \frac{dT_n}{dt} = g_n^{-1} I_n^b \ , \tag{1}$$

where T_n is the temperature of the nth fuse, C its specific heat, g_n its conductance, I_n the current flowing in that fuse. A fuse breaks down, becoming an insulator, when its temperature T_n reaches a given threshold, the same for all. The heterogeneity is on the conductances g_n of the fuses, distributed according to a uniform distribution in the interval $[1 - \Delta/2; 1 + \Delta/2]$. Δ is the measure of the amplitude of disorder and varies from 0 (no disorder) to 2 (maximum disorder). As a result of the delay and relaxation effects embodied in the dynamics of the heating of each fuse, a wealth of novel behaviors emerge, with fractal cracks and critical behavior in the time domain with exponents being continuous functions of b [14]. The motivation behind this model was to introduce the simplest genuine dynamical rupture model which is still amenable to quasi-static elastic calculations. This is done by coupling the dynamical variable to the elastic fields only at the time of each rupture event. This model has the following equivalent mechanical formulation in mode III (antiplane) elasticity. Currents become forces, potentials become displacements, the temperature of a fuse becomes the damage variable of this element, which when reaching one triggers its breakdown.

In previous investigations [14,19], it was shown that the electric resistivity of the system diverges upon approaching the global rupture occurring at time t_r as

$$R \sim (t_r - t)^{-\alpha} \ , \tag{2}$$

with an exponent $1 \leq \alpha(b) \leq 2.3$ in 2D as b decreases from $+\infty$ to 0. It was also observed qualitatively that this power law is observed in a "critical region" that shrinked as the disorder amplitude Δ decreases. The question is whether the critical region vanishes at a finite non-zero value of Δ or just shrinks continuously with Δ, and if so, how?

We study the resistivity $R_{L,\Delta}(t)$ as the observable defined for a system of size L with disorder amplitude Δ at time t. Building on the critical rupture hypothesis, we test whether $R_{L,\Delta}(t)$ can be represented as homogeneous function. From the Π theorem for homogeneous functions [20], we write

$$R_{L,\Delta}(t) = [\frac{\tau}{\Delta^\delta}]^{-\alpha} G\left(\frac{\tau}{\Delta^\delta}, \tau L^{\frac{1}{\nu}} \right) \tag{3}$$

where $\tau \equiv \frac{t_r - t}{t_r}$. The scaling function $G(x, y)$ should scale as

- $L^{-1/\nu} << \tau << \Delta^{\delta}$ (corresponding to $t \to t_r$ with Δ fixed and finite size effects not important). This leads to $G(x \to 0, y \to \infty) \to constant$, i.e. $R \sim [\frac{\tau}{\Delta^{\delta}}]^{-\alpha}$;
- $L^{-1/\nu} << \tau$ and $\Delta^{\delta} \leq \tau$ (corresponding to $\Delta \to 0$ with t fixed and finite size effects not important). This leads to $G(x \to \infty, y \to \infty) \to x^{\alpha}$, i.e. $R \sim constant$;
- $\Delta^{\delta} \leq \tau \leq L^{-1/\nu}$ (corresponding to $\Delta \to 0$ with t fixed and finite size effects important). This leads to $G(x \to \infty, y \to 0) \to y^{\alpha}$, i.e. $R \sim L^{\frac{\alpha}{\nu}}$.

We have introduced the usual finite size scaling ansatz with a characteristic width of the transition scaling as $L^{-1/\nu}$, where ν is the correlation length exponent. We have also assumed that the disorder amplitude Δ determines the size of the region over which the fracturing stays critical and have introduced the disorder exponent δ which may a priori depend on b.

The finite size scaling with the system size L has been previously tested successfully [12–15]. We turn to a test of Eq (3) with respect to the disorder dependence. Figures 1 and 2 show the scaling function G defined from Eq (3) as a function of x for two different values of the exponent b with various values of the disorder field Δ. The data points are obtained by sampling over 25 independent simulations on a square lattice of size 180×180 tilted at 45°, using the method of [14]. These results thus correspond to a significant computational effort. The exponent α was chosen $\alpha(0.5) = 0.9$ and $\alpha(2) = 0.3$ respectively, in accordance to the power law fits done previously [14]. We found that data collapse was optimal using $\delta = 1$ for both values of α.

Figures 2, for $\alpha(b = 2) = 0.3$, show an excellent verification of the scaling relation Eq (3) with an almost complete overlap for the various values of disorder Δ from 0.1 to 0.8. As expected $G(x) \propto x^{\alpha}$ for large x (large $t_r - t$), and crosses over to a constant for small x (t close to t_r). The data for the largest disorder $\Delta = 1.6$ departs from the master curve : this can be attributed to the fact that the resistivity is modified by the pre-existing heterogeneity, even in absence of damage.

For $\alpha(b = 0.5) = 0.9$, simulations with different disorder collapse neatly onto one scaling curve for $x > 5\ 10^{-3}$, going to the expected powerlaw $G(x) \propto x^{\alpha}$ for the largest x and a constant in the region $5\ 10^{-3} \leq x \leq 2\ 10^{-2}$. For smaller values of x, the condition $L^{-1/\nu} << \tau << \Delta^{\delta}$, for which G is a constant and finite size effects are not felt, is not obeyed anymore and is replaced by $L^{-1/\nu} << \tau << \Delta^{\delta}$, for which $g(x \to 0, y \to 0) \to y^{\alpha} \propto x^{\alpha}$. We thus expect and observe indeed that the scaling function G should return to a power law for the smallest values of x after the intermediate plateau. This finite size effect is all the strongest for the smallest b, since smaller values of b give a more diffuse network of cracks [14], whose fractal dimension increases with b, from the value 1 up to the percolation cluster dimension ≈ 1.9 in 2D. As for figure 2, the data for the largest disorder $\Delta = 1.6$ departs from the master curve, for the same reason.

There is a simple explanation for the universal value of the disorder exponent $\delta = 1$, which does not seem to depend on b while the other exponents (the resistivity exponent α and the crack fractal dimensions) do [8]. This has to do with dimensional analysis of (1). Indeed, (1) is a scale invariant equation and since the electric-elastic equations are also scale invariant, dimensional analysis should give the correct scaling law in the time domain. Eq. (1) indicates that the time scales linearly with g. Therefore, the spread in time scales scale with the width of the distribution of g. This leads to the prediction that the width of the critical domain for rupture must scale linearly with the disorder amplitude Δ, hence $\delta = 1$, irrespective of the value of the exponent b. A similar argument gives that $t_r \sim I_{tot}^{-b}$ [14], where I_{tot} is the total current applied to the system. As already pointed out, the thermal fuse model exhibits a simplification in the coupling between the damage field (temperature) and the elastic field (electric current), which occurs only at the time of fuse breaking. This is the origin of this remarkable universal scaling law as a function of the disorder.

IV. CONCLUSION

We have shown that, in a simple dynamical model of rupture in heterogeneous media incorporating long-range elastic forces, scaling holds as a function of the amplitude of the disorder : the width of the critical region is linearly proportional to the amplitude of the disorder. As a consequence, we conclude that, at least in this prototype model, dynamical rupture in random media is a genuine critical phenomenon that occurs for any non-zero disorder however small. In pratice however, the region of control parameter over which the critical regime can be observed becomes exceedingly small for small disorders and may thus mislead to the conclusion of an abrupt behavior more similar to

a first-order transition. This situation is different from the one of bounded load transfer previously investigated [8] in which there is a finite disorder amplitude below which the critical regime disappears and is replaced by an abrupt first-order regime. It would be worthwhile to test these behaviors on other models and in experiments to confirm and extent the present classification.

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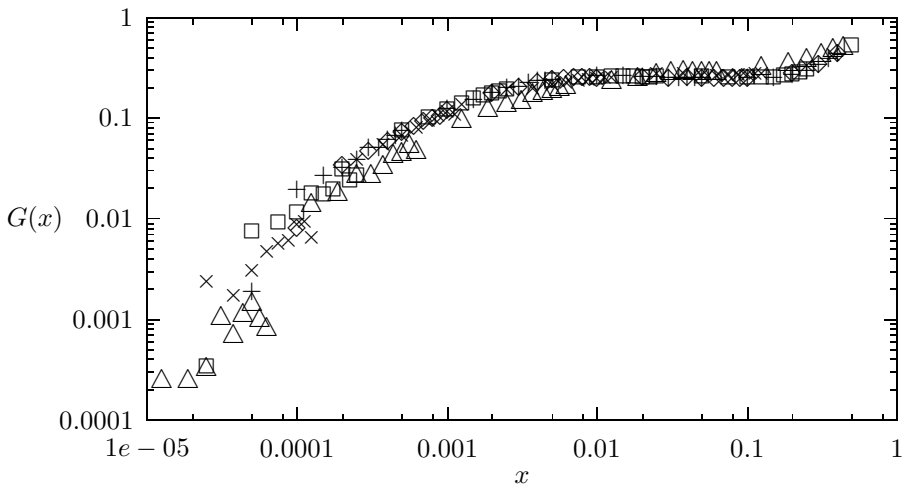

FIG. 1. Scaling function $G(x) = R(x, y)\ x^{\alpha(b)}$ versus x for $b = 0.5$, using $\alpha(0.5) = 0.9$ as obtained independently from [14]. Disorder amplitude $\Delta = 0.1(\diamond)$, $0.2(+)$, $0.4\square$, $0.8(\times)$, and $1.6(\triangle)$. The best overlap was obtained using $\delta = 1$.

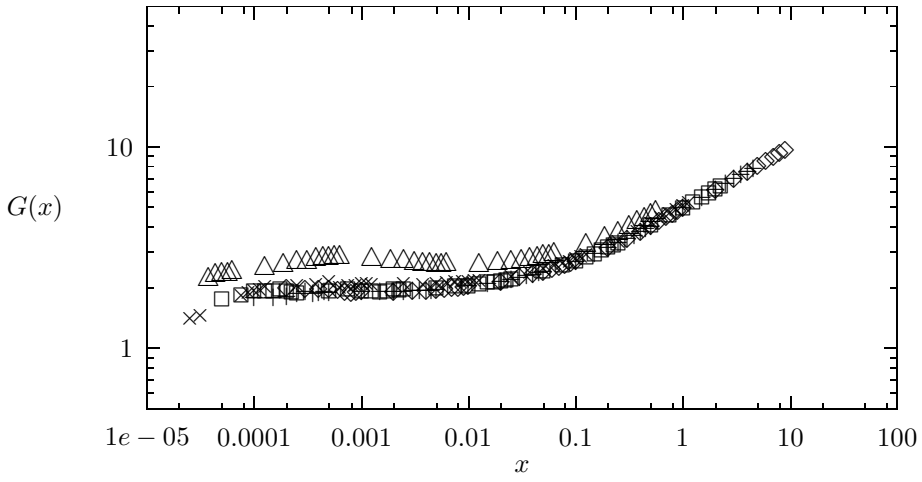

FIG. 2. Scaling function $G(x) = R(x, y)\ x^{\alpha(b)}$ versus x for $b = 2$, using $\alpha(2) = 0.3$ as obtained independently from [14]. Disorder amplitude $\Delta = 0.1(\diamond)$, $0.2(+)$, $0.4\square$, $0.8(\times)$, and $1.6(\triangle)$. The best overlap was obtained using $\delta = 1$.